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LETTER TO THE EDITOR

Free bosons in a scaled external potential

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Abstract. We introduce a suitable set of volume-dependent coherent states and use correlation inequalities to treat the condensation problem of bosons in an external field.

It is well known (Hohenberg 1967) that free bosons in one and two dimensions do not exhibit Bose condensation. However, recently van den Berg (1980) and van den Berg and Lewis (1981) (see also Pulè 1981) proved the existence of Bose condensation in one (and more) dimension(s) for a system of non-interacting bosons moving in an external field given by a potential of the form $V(x/L)=c|x/L|^{\alpha}$ where c and α are positive numbers, and L is the length of the box.

As in any quantum mechanical problem, the fundamental observables are position x and momentum p, satisfying $[x,p]=\mathrm{i}\hbar$. The main point of this Letter is our remark that free Bose systems in an external field, scaled as above, are described by the fundamental observables $x_L=x/L$ and p, hence satisfying $[x_L,p]=\mathrm{i}\hbar/L$. The thermodynamic limit, when L tends to infinity, formally reduces to the classical limit where $\hbar$ tends to zero. As is known (Hepp 1974), the classical limit is made precise by describing the particles in coherent states. Our main contribution consists in stating the suitable form of the coherent states appropriate for the problem and applying correlation inequalities as a method of solution.

We consider the following one-dimensional, length parameter dependent model of identical bosons on the Fock space $\mathscr{H}_{\mathrm{F}}$. The dynamical system is specified by the automorphism groups α_t^L given by

$$\alpha_t^L a^*(\phi)=a^*(\mathrm{e}^{\mathrm{i}th_L}\phi), \qquad \phi\in\mathscr{S}(\mathbb{R}), \tag{1}$$

acting on the Fock creation and annihilation operators a^* and a, where h_L is the one-particle Hamiltonian

$$h_L=-\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2+V(x/L)-\mu_L \tag{2}$$

on the Schwartz space $\mathscr{S}(\mathbb{R})$. The external potential V is a positive continuous function, with global minima zero, and further specified such that $Vf\in\mathscr{S}(\mathbb{R})$ for every $f\in\mathscr{S}(\mathbb{R})$ and h_L is essentially self-adjoint and $\exp(-\beta h_L)$ $(\beta>0)$ is trace-class on $\mathscr{L}_2(\mathbb{R})$ (see Davies 1973).

We denote by ω_L the Gibbs state with the constraint $\omega_L(N)=2L\rho$, by which the local chemical potential μ_L is defined, and where ρ is the fixed density; N is the number operator.

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We show the existence of Bose condensation in one dimension for a general class of potentials, including the choice of van den Berg and Lewis (1981) for $\alpha > \frac{1}{2}$, by investigating the two-point function using correlation inequalities.

We define a family of coherent states as follows: for each $k, q \in \mathbb{R}$

$$f^L_{k,q}(x) = e^{ikx}(\tfrac{1}{2}L^{-\varepsilon})^{1/2} \exp(-\tfrac{1}{2}L^{-\varepsilon}|x - Lq|), \qquad x \in \mathbb{R}, \tag{3}$$

where $0 < \varepsilon < 1$ and $\|f^L_{k,q}\|_2 = 1$. To be precise, we have to modify the function in a sufficiently small neighbourhood of the point $x = Lq$ in order that $f^L_{k,q} \in \mathscr{S}(\mathbb{R})$ and

$$(f^L_{k,q}, h_L f^L_{k,q}) = \tfrac{1}{2}k^2 - \mu_L + V(q) + o(1/L). \tag{4}$$

Hereby $o(1/L)$ tends to zero if L tends to infinity. We can replace $o(1/L)$ by $L^{-\varepsilon}o(1/L)$ for $\varepsilon < \alpha$ if V is Hölder continuous of some order $\alpha > 0$.

For the observable

$$N^L_{k,q} = a^*(f^L_{k,q})a(f^L_{k,q}) \tag{5}$$

we give lower and upper bounds to $\omega_L(N^L_{k,q})$ by applying correlation inequalities. For each $k, q \in \mathbb{R}$ with $k \neq 0$, and for β being the inverse temperature, we obtain

$$\omega_L(N^L_{k,q}) \geq [\![\exp\{\beta[\tfrac{1}{2}k^2 - \mu_L + V(q) + o(1/L)]\} - 1]\!]^{-1} \tag{6}$$

by applying the correlation inequality (Fannes and Verbeure 1977) to $a(f^L_{k,q})$,

$$\omega_L([a^*(f^L_{k,q}), a(f^L_{k,q})]) \ln[\omega_L(N^L_{k,q})/1 + \omega_L(N^L_{k,q})] \leq \beta\omega_L([a(f^L_{k,q}), a^*(h_L f^L_{k,q})]), \tag{7}$$

and equation (4). It follows from (7) that $\mu_L \leq 0$.

To get an upper bound we restrict the potential to be Hölder continuous of order $\alpha > \frac{1}{2}$. In this case for each $k, q \in \mathbb{R}$, $k \neq 0$,

$$\omega_L(N^L_{k,q}) \leq [\![\exp\beta\{\tfrac{1}{2}k^2 + V(q) - \mu_L - L^{-\varepsilon}[|o(1/L)| + |o(1/L)|\omega_L(N^L_{k,q})^{-1/2}]\} - 1]\!]^{-1} \tag{8}$$

if $\varepsilon < (2\alpha - 1)/(2\alpha + 1)$. This follows from the correlation inequality (Fannes and Verbeure 1977)

$$-\beta\omega_L(a^*(f^L_{k,q})a(h_L f^L_{k,q})) \geq \omega_L(N^L_{k,q}) \ln[\omega_L(N^L_{k,q})/1 + \omega_L(N^L_{k,q})] \tag{9}$$

and the estimate

$$\omega_L(a^*(f^L_{k,q})a(g^L_{k,q})) = L^{-\varepsilon}o(1/L)\omega_L(N^L_{k,q})^{1/2}, \tag{10}$$

where

$$g^L_{k,q}(x) = (V(x/L) - V(q))f^L_{k,q}(x). \tag{11}$$

To prove (10) we apply the Cauchy–Schwarz inequality twice after expanding $g^L_{k,q}$ into an orthonormal basis of $\mathscr{L}_2(\mathbb{R})$ and using the relation $\omega_L(N) = 2L\rho$. This yields

$$|\omega_L(a^*(f^L_{k,q})a(g^L_{k,q}))|^2 \leq 2L\rho\|g^L_{k,q}\|^2\omega_L(N^L_{k,q}). \tag{12}$$

Finally, to find a bound for $\|g^L_{k,q}\|^2$ we decompose the integration into an $\varepsilon_L = \varepsilon_0 L^{\delta+\varepsilon-1}$ $(\varepsilon_0 > 0, \delta > 0)$-neighbourhood and its complement, $\|g^L_{k,q}\|^2 = G_1 + G_2$, where δ can be chosen sufficiently small such that

$$G_1 \sim \varepsilon_0^2 L^{2\alpha(\delta+\varepsilon-1)} = \varepsilon_0^2 L^{-1-\varepsilon}o(1/L). \tag{13}$$

The contribution G_2 from the complement decays exponentially for large L, because $Vf \in \mathscr{S}(\mathbb{R})$ for any $f \in \mathscr{S}(\mathbb{R})$. Other terms on the left-hand side of (9) coming from the modification of (3) at $x = Lq$ are again of the type $L^{-\varepsilon}(o(1/L) + o(1/L)\omega_L(N^L_{k,q})^{1/2})$.

Next we study the off-diagonal part of the two-point function. By applying the Bogoliubov inequality

$$|\omega_L([A, B])|^2 \leqslant \tfrac{1}{2}\beta\omega_L(A^*A + AA^*)\omega_L([B^*, [H_L, B]]) \tag{14}$$

with $A = a^*(f^L_{k,q})a(f^L_{k',q'})$ and $B = a^*(f^L_{k,q})a(f^L_{k,q})$ we obtain

$$\begin{aligned} |\omega_L(N^L_{k,q})(f^L_{k',q'}, f^L_{k,q}) - \omega_L(a^*(f^L_{k,q})a(f^L_{k',q'}))|^2 \\ \leqslant \tfrac{1}{2}\beta[4\omega_L(N^L_{k,q})\omega_L(N^L_{k',q'}) + \omega_L(N^L_{k,q}) \\ + \omega_L(N^L_{k',q'})]\{2\omega_L(N^L_{k,q})(f^L_{k,q}, h_L f^L_{k,q}) \\ - \omega_L(a^*(h_L f^L_{k,q})a(f^L_{k,q})) - \omega_L(a^*(f^L_{k,q})a(h_L f^L_{k,q}))\}. \end{aligned} \tag{15}$$

The term in braces on the right-hand side is bounded by $L^{-\varepsilon}\mathrm{o}(1/L)$ uniformly in k and q outside $k=0$. From (8), $\omega_L(N^L_{k,q})$ is bounded above by $(\frac{1}{2}k^2 + V(q) - \mu)^{-1}$ for large k. Repeating the estimations which led to (8) by using now $a(h_L f^L_{k,q} - (\frac{1}{2}k^2 + V(q) - \mu - \frac{1}{8}L^{-2\varepsilon})f^L_{k,q})$ instead of $a(f^L_{k,q})$ in the correlation inequality (9), and substituting the resulting bound in (9), we can improve the upper bound of $\omega_L(N^L_{k,q})$ to become $(\frac{1}{2}k^2 + V(q) - \mu)^{-2}$ for large k. Therefore the square root of the right-hand side of (15) is integrable with respect to k at infinity, and to q if V grows fast enough for large $|q|$. This condition on V might be eliminated after an infinite repetition of the above iteration. Expanding the orthonormal basis by which N is described into our coherent states and using the completeness relation

$$(f, f) = L\int \frac{\mathrm{d}k}{2\pi}\mathrm{d}q|(f, f^L_{k,q})|^2 \tag{16}$$

$(L>0, f \in \mathscr{L}_2(\mathbb{R}))$, we obtain

$$\frac{1}{2L}\omega_L(N) = \frac{1}{2}\int \frac{\mathrm{d}k}{2\pi}\mathrm{d}q\, \omega_L(N^L_{k,q}) + R_L, \tag{17}$$

where according to (15) and the remarks made thereafter R_L vanishes in the limit $L \to \infty$. This implies, for instance for fixed $\mu<0$, with (6) and (8),

$$\lim_{L\to\infty}\frac{1}{2L}\omega_L(N) = \frac{1}{2}\int \frac{\mathrm{d}k}{2\pi}\mathrm{d}q\{\exp[\beta(\tfrac{1}{2}k^2 + V(q) - \mu)] - 1\}^{-1} =: \rho_\beta(\mu) \tag{18}$$

uniformly in μ. Since $\rho_\beta(\mu)$ is strictly monotonic in μ, it follows from standard arguments (Davies 1973, Landau and Wilde 1979) that μ_L converges to some μ, which is less than zero if $\rho < \rho_\beta(0)$, and equal to zero if $\rho \geqslant \rho_\beta(0)$.

In this one-dimensional Bose gas, condensation occurs due to the presence of an external potential which allows $\rho_\beta(0) < \infty$. If the potential takes its absolute minima only at a finite number n of positions, then the density of the condensate is

$$\rho_0 = \lim_{\varepsilon\to 0}\lim_{L\to\infty}\sum_{i,j=1}^{n}\int_{\varepsilon_i}\mathrm{d}q\int_{\varepsilon_j}\mathrm{d}q'\int_{\varepsilon}\frac{\mathrm{d}k\,\mathrm{d}k'}{4\pi^2}\tfrac{1}{2}L(f^L_{k',q'}, f^L_{k,q})\omega_L(a^*(f^L_{k,q})a(f^L_{k',q'})), \tag{19}$$

where the integrations are on ε-neighbourhoods of the positions of the absolute minima of V or of $k=0$. If we expand the total density $\rho = (1/2L)\omega_L(N)$ again into the coherent states $f^L_{k,q}$ and decompose the integrations into the neighbourhoods and their complements, we observe that the mixed expressions tend to zero after taking the two limits as in (19). With the technique by which (17) was handled we arrive at the bound

$$\rho - \rho_0 \leq \rho_\beta(0), \tag{20}$$

which implies that the condensate density ρ_0 becomes strictly positive for sufficiently large β, if the potential V is chosen such that $\rho_\beta(0) < \infty$. It is clear from (19) that the external potential forces the bosons to condensate into the absolute minima with vanishing momenta. However, with our method we are unable to calculate the relative occupation over the minima.

References

van den Berg M 1980 *Phys. Lett.* **78A** 88
van den Berg M and Lewis J T 1981 *Commun. Math. Phys.* **81** 475
Davies E B 1973 *Commun. Math. Phys.* **30** 229
Fannes M and Verbeure A 1977 *Commun. Math. Phys.* **57** 165
Hepp K 1974 *Commun. Math. Phys.* **35** 265
Hohenberg P C 1967 *Phys. Rev.* **158** 383
Landau L J and Wilde I F 1979 *Commun. Math. Phys.* **70** 43
Pulè J V 1981 *The Free Boson Gas in a Weak External Potential, Preprint,* to appear in *J. Math. Phys.*